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# Some connections between global hyperbolicity and geodesic completeness

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**Abstract.** *We establish for space-times obeying certain curvature conditions (consistent with gravity being attractive) some clear cut connections between global hyperbolicity and timelike geodesic completeness. We show, under suitable circumstances, that if the future of a spacelike hypersurface is future timelike geodesically complete then it is global hyperbolic. A partial converse is also obtained. One of our results is a consequence of a «splitting theorem» for space-times which admit a maximal hypersurface. Our main results are used to improve certain aspects of some splitting theorems previously obtained in the literature.*

## INTRODUCTION

To the Riemannian geometer, no assumption is more natural than that of geodesic completeness. However, as is well-known, the concept of geodesic completeness has a very different status in Lorentzian geometry. For one thing, many of the space-times of physical interest are nonspacelike geodesically incomplete. For another, the naive analogue of the Hopf-Rinow theorem does not hold. The standard condition which insures that two timelike related points in a Lorentzian manifold can be joined by a maximal timelike geodesic is not geodesic completeness, but global hyperbolicity. In general, in the absence of any additional condition on space-time, there is no relationship between these two concepts. As a simple illustration, consider the space-time obtained by removing the origin from Minkowski space. By making a suitable conformal change in the metric

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near the origin we obtain a space-time which is geodesically complete but not globally hyperbolic. However, in making the conformal change we have altered the curvature. The main purpose of this paper is to demonstrate that for space-times which obey certain sectional or Ricci curvature conditions (consistent with the fact that gravity is attractive), there are some clear cut connections between global hyperbolicity and timelike geodesic completeness.

Before proceeding to a detailed description of the main results, we make a few remarks concerning their general nature. Space-times obeying the kind of curvature conditions we shall impose tend to be nonspacelike geodesically incomplete, either to the past or the future. For this reason, we shall consider space-times which are complete «in one direction» (past of future), but perhaps incomplete in the other. Moreover, we shall not impose curvature conditions, like the so-called generic condition, which require some curvature object to obey a *strict* inequality; only weak curvature inequalities will be considered. Theorems A and B, stated further on in the introduction, show that, under suitable circumstances, if the future of a spacelike hypersurface is future timelike geodesically complete then it is globally hyperbolic. Theorem B is a consequence of Theorem C, which is a «splitting theorem» for space-times which admit a maximal spacelike hypersurface. Theorem D is a partial converse to Theorem A. Global hyperbolicity does not imply geodesic completeness even under favorable curvature conditions (consider a horizontal strip in Minkowski 2-space). However, Theorem D shows that, under suitable circumstances, if the future of a spacelike hypersurface is globally hyperbolic then it will be future timelike geodesically complete, provided, for example, at least one timelike geodesic is future complete.

We now give a more detailed account of these results. For causal theoretic notions used, but not defined below, we refer the reader to Hawking and Ellis [HE].

By a *space-time* we mean a $C^\infty$ connected Hausdorff manifold $M$ ($\dim M \geqslant 2$), equipped with a $C^\infty$ metric $g$ of Lorentzian signature $(-, +, \ldots +)$, with respect to which $M$ is time orientable. We define a *spacelike hypersurface* in $M$ to be a connected subset $S \subset M$ with the property that for each $p \in S$, there is a neighborhood $U$ of $p$ such that $S$ is acausal and edgeless $U$. $S$ is necessarily an embedded topological submanifold of $M$ of co-dimension one. A *smooth spacelike hypersurface* is a connected $C^\infty$ embedded codimension one submanifold of $M$ with everywhere timelike normal. (Those readers, who prefer, may assume all spacelike hypersurfaces are smooth). We shall make use of the concept of *future causal completeness* introduced in [G2]. A spacelike hypersurface $S$ is said to be future causally complete if and only if for each $p \in J^+(S)$, the closure in $S$ of $J^-(p) \cap S$ is compact. Physically, $S$ is future causally complete provided the information from $S$ that reaches an observer at any instant comes from a finite nonsingular region in $S$. (We refer the reader to [G2] for a more detailed discussion of the physical significance of this condition). A future causally complete spacelike hypersurface is necessarily a closed subset of space-time. We note that *compact* spacelike hypersurfaces and Cauchy surfaces (defined below) are future causally complete.

In this paragraph we recall some basic definitions and results from causal theory. By definition, a space-time $M$ is globally hyperbolic if and only if $M$ is strongly causal and the causal intervals $J^+(p) \cap J^-(q)$ are compact for all $p$ and $q$ in $M$. A *Cauchy surface* in $M$ is an acausal spacelike hypersurface which intersects every inextendible nonspacelike curve. A basic result of causal theory asserts that $M$ is globally hyperbolic if and only if $M$ admits a Cauchy surface. Given an acausal spacelike hypersurface $S$, one can characterize its being Cauchy in terms of its domain of dependence $D(S) = D^+(S) \cup D^-(S)$, or its Cauchy horizon $H(S) = H^+(S) \cup H^-(S)$. An acausal spacelike hypersurface $S$ is Cauchy iff $D(S) = M$ iff $H(S) = \phi$. In order to express our results in purely «futuristic» terms we introduce the following terminology. We will say that an acausal spacelike hypersurface $S$ is a *future Cauchy surface* if and only if its future Cauchy horizon is trivial, $H^+(S) = \phi$. A future Cauchy surface $S$ separates $M$ into two components, one of which is the chronological future of $S$, $I^+(S)$. Moreover, since, for a future Cauchy surface $S$, $J^+(S) = D^+(S)$, $I_+(S)$ is globally hyperbolic.

To conclude the preliminaries, we make some remarks about our curvature conditions. $M$ is said to obey the strong energy condition if and only if $\mathrm{Ric}(X, X) \geqslant 0$ for all timelike vectors $X$ in $M$. The Ricci curvature of a unit timelike vector can be written as minus the sum of tidal accelerations, or in geometric terms, as *minus* the sum of timelike sectional curvatures. In some of our results we require $M$ to obey the strong condition, while in others we impose the more stringent requirement that $M$ have nonpositive timelike sectional curvatures. The Friedmann cosmological models, and perturbations of them, have nonpositive timelike sectional curvatures.

The remainder of the introduction is devoted to stating the main results of the paper. Our first result establishes circumstances under which the future completeness of $I^+(S)$ implies the global hyperbolicity of $I^+(S)$.

THEOREM A. *Let $M$ be a space-time with non-positive timelike sectional curvatures. Let $S$ be an acausal future causally complete spacelike hypersurface in $M$. Then, if $J^+(S)$ is future timelike geodesically complete, $S$ is a future Cauchy surface, $H^+(S) = \phi$.*

Modulo known topological obstructions, asymptotically flat space-times typically admit maximal (i.e. mean curvature zero) hypersurfaces. If one assumes in Theorem A that $S$ is maximal then the sectional curvature condition can be replaced by the strong energy condition.

THEOREM B. *Let $M$ be a space-time wich obeys the strong energy condition,* $\mathrm{Ric}(X, X) \geq 0$ *for all timelike vectors $X$. Suppose $S$ is a smooth acausal future causally complete spacelike hypersurface in $M$ which is maximal. Then, if $J^+(S)$ is future timelike geodesically complete, $S$ is a future Cauchy surface.*

Examples such as anti-de Sitter space and the Reissner-Nordstrom solution illustrate the importance of the future causal completeness assumption in Theorems A and B. Theorem B is a consequence of the following *splitting* theorem for space-times with maximal hypersurfaces, which is of some interest in its own right.

THEOREM C. *Let $M$ be a space-time which obeys the strong energy condition,* $\mathrm{Ric}(X, X) \geq 0$ *for $X$ timelike. Suppose $M$ contains a smooth acausal maximal spacelike hypersurface $S$, which is either geodesically complete or future causally complete. Assume $J^+(S)$ is future timelike geodesically complete. If $\gamma$ is a future complete $S$-ray such that $I^-(\gamma) \cap J^+(S)$ is globally hyperbolic then $J^+(S)$ is isometric to* $([0, \infty) \times S, -dt^2 \oplus h)$, *where $h$ is the induced metric on $S$.*

By an *S-ray* we mean a future inextendible geodesic $\gamma$ emanating from a point in $S$ such that $\gamma$ realizes the distance to $S$ from each of its points. Eschenburg [E], Newmann [N] and the author [G3] have obtained splitting theorems for space-times which obey the strong energy condition and contain a complete timelike line. The proof of Theorem C involves a variation of arguments used in [E] and [G3]. As consequences of our main results, we obtain some other space-time splitting results Theorems 2.5 and 2.6, Corollary 3.4) wich improve certain aspects of splitting theorems previously obtained in the literature (see [B], [B + ]). Such results may be interpreted as «rigid» singularity theorems; for discussions of this point see [B], [G1], [G3], or the review article [G4].

The next theorem is a partial converse to Theorem A. It shows that global hyperbolicity together with a little bit of completeness can imply a lot of completeness.

THEOREM D. *Let $M$ have nonpositive timelike sectional curvatures and contain a compact future Cauchy surface $S$. Then, if*

(1) $$\sup_{x \in M} d(S, x) = \infty,$$

*where $d$ is the Lorentzian distance function, $J^+(S)$ is future timelike* *geodesically complete.*

Simple example show that the assumption of *compactness* in Theorem D can not be replaced by *future causal completeness.* It is natural to consider to what extent Theorems A and D remain valid when the sectional curvature condition is replaced by the strong energy condition. A recent example of Bartnik [B] shows that Theorem D with this weaker Ricci curvature condition does not hold.

In Section 2 and 3 we present the proofs of Theorems A and D, and Theores B and C, respectively, and derive some consequences. We first took up this issue of the relationship between global hyperbolicity and completeness in [G2]. Some of the results obtained in the present paper can be used to improve some of the results in [G2].

## 2. SPACE-TIMES WITH NONPOSITIVE TIMELIKE SECTIONAL CURVATURES

We will have occasion in this section and the next to make use of the following lemma which is proved in Galloway [G2].

LEMMA 2.1. *Let $M$ be a space-time which admits an acausal future causally complete spacelike hypersurface $S$. If $H^+(S) \neq \phi$ then for each $p \in H^+(S)$ there exists a future inextendible $S$-ray $\gamma$ contained in $D^+(S) \cap I^+(p)$.* ∎

The principal geometric tool upon which the proofs of Theorems A and D rely is the Lorentzian version of the triangle comparison theorem due to Harris [H1]. Below we state the form of Harris' theorem that will be needed here. We say that $(\gamma_1, \gamma_2, \gamma_3)$ is a timelike geodesic triangle provided $\gamma_1, \gamma_2, \gamma_3$ are future directed timelike geodesic segments such that $\gamma_2$ extends from the past end point of $\gamma_1$ to the future end point of $\gamma_3$, and the future end point of $\gamma_1$ coincides with the past end of point of $\gamma_3$. Let $\alpha_3 := \langle \gamma_1'(0), \gamma_2'(0) \rangle$ and $\alpha_2 := \langle -\gamma_1'(L_1), \gamma_3' \rangle$ be the «angles» between $\gamma_1$ and $\gamma_2$, and $\gamma_1$ and $\gamma_3$, respectively.

LEMMA 2.2 *(Harris). Let $M$ be a globally hyperbolic space-time with nonpositive timelike sectional curvatures. For any timelike geodesic triangle $(\gamma_1, \gamma_2, \gamma_3)$ in $M$, with $\gamma_2$ and $\gamma_3$ maximal, there exists a correspondig timelike geodesic triangle in Minkowski space $(\overline{\gamma}_1, \overline{\gamma}_2, \overline{\gamma}_3)$ such that $L(\overline{\gamma}_i) = L(\gamma_i)$, $i = 1, 2, 3$* ($L$ = length), *$\overline{\alpha}_2 \geqslant \overline{\alpha}_2$, and $|\overline{\alpha}_3| \leqslant \alpha_3$.* ∎

We will need to make use of (a slight extention of) the concept of a co-ray which was introduced in Beem et al [B+]. Let $\gamma : [0, a) \to M$ be a future inextendible timelike ray. Given any $x \in I^-(\gamma)$, let $\{x_n\}$ be a sequence of points in $M$ converging to $x$, and let $\{s_n\}$ be a sequence of numbers increasing to $a$. For all $n$ sufficiently large we will have $x_n \ll \gamma(s_n)$. For all such $n$, let $\{\eta_n\}$ be any maximal timelike geodesic segment from $x_n$ to $\gamma(s_n)$. A future co-ray $\eta$ to $\gamma$ is, by definition, any limit curve of the sequence $\{\eta_n\}$. In the speciel case in which $x_n = x$ for all $n$, $\eta$ is called an asymptote of $\gamma$. A co-ray is necessarily a future inextendible maximal length timelike or null geodesic. We say that the timelike co-ray condition holds on $I^-(\gamma) \cap I^+(r)$ $(r = \gamma(0))$ provded for all $x \in I^-(\gamma) \cap I^+(r)$, every co-ray to $\gamma$ from $x$ is timelike.

LEMMA 2.3. *Let $M$ be a globally hyperbolic space-time with nonpositive timelike sectional curvatures. If $b$ is a future inextendible timelike ray emanating from $r \in M$ then the timelike co-ray condition holds on $I^-(\gamma) \cap I^+(r)$.*

Lemma 2.3. is proved in Beem et al [B+] in the case $\gamma$ is future complete. One can handle the case in which $\gamma$ is future incomplete by a similar kind of argument, and we omit the details. ∎

The following lemma establishes a basic connection between global hyperbolicity and completeness in space-time of nonpositive timelike sectional curvature.

LEMMA 2.4. *(The completeness lemma). Let $M$ be a space-time with nonpositive timelike sectional curvatures. Let $\gamma$ be a future complete timelike ray in $M$ emanating from $r \in M$, and assume the set $I^+(r) \cap I^-(\gamma)$ is globally hyperbolic. Then $I^+(r) \subset I^-(\gamma)$ and $I^+(r)$ is future timelike geodesically complete.*

*Proof.* It is sufficient to show that $N := I^+(r) \cap I^-(\gamma)$ is future timelike geodesically complete, for then it follows easily that $I^+(r)$ is contained in $I^-(\gamma)$. To this end, let $\eta$ be a future directed timelike geodesic which is future inextendible in $N$. We may assume that $\eta$ has a past end point $x \in N$. We construct an asymptote $\mu$ of $\gamma$ from $x$, which by Lemma 2.3 is a *timelike* geodesic ray contained in $N$. By its construction, there exist numbers $r_k \uparrow \infty$ and maximal geodesic segments $\mu_k$ from $x$ to $\gamma(r_k)$, converging to $\mu$, such that for all $k$,

$$|\langle \mu_k'(0), \eta'(0)\rangle| \leqslant C,$$

for some positive constant C. Since $N$ is globally hyperbolic, $\eta$ cannot be contained in the compact set $J^-(\gamma(r_k), N) \cap J^+(x, N)$. It follows that $\eta$ meets $\partial I^-(\gamma(r_k), N) = J^-(\gamma(r_k), N) - I^-(\gamma(r_k), N)$ at some point $x_k$. Since $d(x_k, \gamma(r_k)) = 0$ for all $k$, the continuity of the Lorentzian distance function implies that there exists an $\epsilon > 0$, and for each k, a point $y_k \in \eta$, $x \ll y_k \ll x_k$, such that $d(y_k, \gamma(r_k)) < \epsilon$.

Let $\eta_k$ be the segment of $\eta$ from $x$ to $y_k$, and $\sigma_k$ be a maximal geodesic segment from $y_k$ to $\gamma(r_k)$. We apply the triangle comparison theorem to $(\eta_k, \mu_k, \sigma_k)$. Let $a_k = L(\eta_k)$, $b_k = L(\sigma_k)$, and $x_k = L(\mu_k)$. By Lemma 2.2 there exists a timelike geodesic triangle $(\overline{\eta}_k, \overline{\mu}_k, \overline{\sigma}_k)$ in Minkowski space such that $L(\overline{\eta}_k) = a_k$, $L(\overline{\mu}_k) = c_k$, $L(\overline{\sigma}_k) = b_k$, and $|\overline{\beta}_k| \leq |\beta_k|$, where $\beta_k := \langle \mu_k'(0), \eta_k'(0)\rangle$ and $\overline{\beta}_k$ are the «angles» opposite $\sigma_k$ and $\overline{\sigma}_k$, respectively. The lax of cosines in Minkowski space gives,

$$(2) \qquad b_k^2 = a_k^2 + c_k^2 + 2a_k c_k \overline{\beta}_k .$$

The sequence $\{\overline{\beta}_k\}$ is bounded by $C$. If $\eta$ is future incomplete then $\{a_k\}$ is bounded. Since $c_k \to \infty$, and $\{a_k\}$ and $\{\overline{\beta}_k\}$ are bounded, (3) implies that $b_k \to \infty$. But the sequence $\{b_k\}$ is bounded by $\epsilon$. Thus, $\eta$ must be future complete, and the lemma is established. ▪

Theorem A is now readily establised.

*Proof of Theorem A.* Suppose $H^+(S) \neq \phi$. Choose a point $q \in H^+(S)$. By Lemma 2.1, there exist a future inextendible timelike geodesic ray $\gamma$ contained in $D^+(S) \cap I^-(q)$ emanating from some point $r \in S$. It follows easily from the definition of the future domain of dependence that $I^+(r) \cap I^-(\gamma)$ is contained in int $D^+(S)$, and hence $I^+(r) \cap I^-(\gamma)$ is globally hyperbolic.

Suppose $\gamma$ is future complete. Then, from Lemma 2.4 we have, $I^+(r) \subset I^-(\gamma)$, which implies in particular that $q \in I^-(\gamma)$. But this contradicts the fact that $I^+(H^+(S))$ and $D^+(S)$ never meet. ▪

The following rigidity result improves Theorem 5.5 in Beem et al. [B$^+$] by removing the assumption of global hyperbolicity.

THEOREM 2.5. *Let $M$ be a noncompact space-time having nonpositive timelike sectional curvatures and containing a compact spacelike hypersurface. Then $M$ is either timelike geodesically incomplete or «splits» isometrically into the product $(\mathbb{R} \times S, -dt^2 \oplus h)$, where $(S, h)$ is a compact Riemannian manifold.*

It is an interesting open problem to determine whether Theorem 2.5 remains valid with the sectional curvature condition replaced by the strong energy condition; see [G4].

*Proof.* Assume $M$ is timelike geodesically complete. If $S$ is acausal, then Theorem A and its time-dual imply that $S$ is Cauchy. That $M$ splits as required now follows from Theorem 5.5 in [B+]. Suppose now that $S$ is not acausal. Introduce the Geroch covering space-time $(\widetilde{M}, \pi)$ (see [HE], p. 205). $\pi^{-1}(S)$ consists of countably many copies $\widetilde{S}_i$ of $S$, each of which is acausal. Theorem A applied to $\widetilde{M}$

shows that each $\widetilde{S}_i$ is Cauchy. It then follows from basic properties of the covering that $\pi(J^+(\widetilde{S}_0) \cap J^-(\widetilde{S}_1)) = M$, and hence that $M$ is compact, a contradiction. ∎

Note the proof actually shows that if $M$ is compact then it is covered by a space-time isometric to $(\mathbb{R} \times S, -dt^2 \oplus h)$. We now proceed to the proof of Theorem D, which requires a little more effort.

*Proof of Theorem D.* The first step in the proof is to construct a future *complete* $S$-ray. Let $\{q_n\}$ be a sequence of points in $M$ such that $d(S, q_n) \to \infty$. Let $\gamma_n$ be a timelike geodesic segment from $r_n \in S$ to $q_n$ which realizes the distance from $S$ to $q_n$. The sequence $\{r_n\}$ has an accumulation point $r \in S$. By standard convergence results (see e.g. Beem and Ehrlich [BE]), there exists a subsequence $\{\gamma_k\}$ of $\{\gamma_n\}$ which converges to a future inextendible causal curve $\gamma$ emanating from $r$. The maximality of the $\gamma_n$'s insures that $\gamma$ is a timelike geodesic $S$-ray. We prove that $\gamma$ is future complete. (Even though $\gamma$ is obtained as a limit of arbitrarily long segments, one can show by example that $\gamma$ need not be complete if one drops either the assumption that $S$ is Cauchy or the assumption of nonnegative timelike sectional curvatures).

For technical reasons we need to perturb $S$ a constant distance to the future. For fixed $\delta > 0$, consider $S' = \{x \in M : d(S, x) = \delta\}$. It is not difficult to show that, for $\delta$ sufficiently small, $S'$ is a compact Cauchy surface. Let $\gamma$ meet $S'$ at $p$. When $k$ is large enough $\gamma_k$ meets $S'$ at $p_k$, say. Let $\mu_k$ be the portion of $\gamma_k$ from $p_k$ to $q_k$, and $\mu$ be the portion of $\gamma$ to the future of $p$. The sequence $\{\mu_k\}$ converges to $\mu$, and $p_k \to p$. Using the reverse triangle inequality and the definition of $S'$, one sees that $\mu_k$ realizes the distance from $S'$ to $q_k$, and hence $\mu$ is an $S'$-ray. Below we will use the fact, insured by Lemma 2.3, that the timelike co-ray condition holds at $p \in I^+(r) \cap I^-(\gamma)$.

When $k$ is large enough, $q_k$ will be in the timelike future of $p$. Thus, as in the proof of the completeness lemma, we can find a sequence of points $\{y_k\}$ along $\mu$, with $y_k \in I^-(q_k)$ such that the sequence $\{d(y_k, q_k)\}$ is bounded. Let $\sigma_k$ be a maximal segment from $y_k$ to $q_k$. Since $\mu$ cannot be contained in the past of any $q_k$ in the future of $p$ (without a strong causality violation), we can choose the $y_k'$s to «exhaust» $\mu$ to the future. In particular, $p_k \ll y_k$ for $k$ sufficiently large. Let $\eta_k$ be a maximal segment from $p_k$ to $y_k$. Since $\mu$ is an $S'$-ray, we have $L(\eta_k) \leqslant L(\mu) < L(\gamma)$. Using the fact that the unit vectors $\mu_k'(0) \to \mu'(0)$ as $k \to \infty$, the timelike co-ray condition at $p$ implies that the angles $\{\langle\mu_k'(0), \eta_k'(0)\rangle\}$ are bounded. By applying the triangle comparison theorem to $\eta_k, \mu_k, \sigma_k$ as in the proof of the completeness lemma, one concludes $L(\sigma_k) = d(y_k, q_k) \to \infty$ if it is assumed that $L(\gamma) < \infty$. Thus $\gamma$ must be future complete.

By the completeness lemma, $I^+(r) \subset I^-(\gamma)$ and $I^+(r)$ is future timelike geodesically complete. Fix a point $v \in I^+(r)$, and consider the achronal boundary

$\partial I^+(v) = J^+(v) - I^+(v)$. The present aim is to show that $\partial I^+(v)$ is a compact achronal Cauchy surface. Since compact achronal boundaries in globally hyperbolic space-times are necessarily Cauchy, it is sufficient to show that $\partial I^+(v)$ is compact. To establish the compactness of $\partial I^+(v)$ it is enough to show that each future inextendible null geodesic $\eta : [0, a) \to M$ emanating from $v$ eventually enters into $I^+(v)$ (see e.g. the proof of Proposition 7.18 in [BE]). To this end we let $w_n = \eta(s_n)$, where $s_n \uparrow a$, and show that $w_n \in I^+(v)$ for some $n$.

The reverse triangle inequality implies $\{d(r, w_n)\}$ is increasing. We show that $d(r, w_n) \uparrow \infty$. Choose a point $w \in I^-(v) \cap I^+(r)$. Let $\alpha_n$ and $\beta_n$ be maximal timelike geodesic segments from $r$ and $w$, respectively, to $w_n$. Let $\sigma$ be a maximal segment of length $L$ from $r$ to $w$. Let $\Theta_n$ be the «angle» between $\beta_n$ and $\sigma$, i.e. $\Theta_n = \langle \beta_n'(0), -\sigma'(L) \rangle$. Let $a_n = L(\alpha_n)$ $(= d(r, w_n))$, and $b_n = L(\beta_n)$. Suppose first that $\{\Theta_n\}$ is bounded. Then $\{\beta_n'(0)\}$ lies in a compact subset of the unit timelike bundle. The $\beta_n's$ cannot be bounded in length, for otherwise their end points $\{w_n\}$ would lie in a compact subset of $M$ which would imply that $\eta$ is imprisoned in a compact set. Thus the sequence $\{b_n\}$ is unbounded. The reverse triangle inequality now implies that $a_n = d(r, w_n) \uparrow \infty$, as was to be shown. Now suppose that $\{\Theta_n\}$ is unbounded. By taking a subsequence, we may assume $\Theta_n \uparrow \infty$. Applying the triangle oomparison theorem in conjunction with the law of cosines gives,

$$a_n^2 = L^2 + b_n^2 + 2Lb_n\overline{\Theta}_n,$$

where $\overline{\Theta}_n \geqslant \Theta_n$. The reverse triangle inequality implies that the sequence $\{b_n\}$ is bounded below by $d(w, v)$. Hence, from the equation above we see that $a_n \uparrow \infty$.

We have established that $d(r, w_n) \uparrow \infty$ as $n \uparrow \infty$, from which it immediately follows that $d(S, w_n) \uparrow \infty$. Let $\eta_n$ be a timelike geodesic segment from $x_n \in S$ to $w_n$ which realizes the distance from $S$ to $w_n$. Let $x \in S$ be an accumulation point of $x_n$. As in the beginning of the proof, there exists a subsequence $\eta_k$ of $\eta_n$ which converges to a future complete $S$-ray $\eta$. Choose a point $y \in I^+(x) \cap \cap I^+(v)$. (Since $w_k \in I^+(x)$ for all $k$ sufficiently large and $I^+(w_k) \subset I^+(v)$ this set is nonempty). According to Theorem 2 in [H 2], we know that $\eta$ enters $I^+(y) \subset I^+(v)$. But then it follows that $\eta_k$ enters $I^+(v)$ for $k$ sufficiently large, and hence $w_k \in I^+(v)$ for such $k$.

We have shown that each future inextendible null geodesic emanating from $v \in I^+(r)$ enters $I^+(v)$. As discussed above, this implies that $\partial I^+(v)$ is a compact achronal Cauchy hypersurface. The proof of Theorem D is now easily completed. Let $\sigma$ bo a future inextendible timelike geodesic in $M$. Since $\partial I^+(v)$ is Cauchy, $\sigma$, if it does not start in $I^+(v)$, eventually enters $I^+(v)$, and hence is contained in $I^+(r)$ from some point on. Since $I^+(r)$ is future timelike geodesically complete, $\sigma$ is future complete. ∎

Theorem D immediately yields the following improvement of Theorem 5.5 in [B+].

THEOREM 2.6. *Let M have nonpositive timelike sectional curvatures and contain a compact Cauchy surface S. Then we have,*

$$\sup_{x\in J^+(S)} d(S, x) < \infty \quad \textit{or} \quad \sup_{x\in J^-(S)} d(x, S) < \infty,$$

*or else M splits as in Theorem 2.5.* ∎

This theorem expresses in a more general context the totality of incompleteness evidenced in the Friedmann models.

## 3. SPACE-TIMES WITH MAXIMAL HYPERSURFACES

We begin this section with the proof of Theorem C. Let the setting be as in the statement of Theorem C in the introduction. Let $N$ be the future directed unit normal field along $S$. Let $E : [0, \infty) \times S \to J^+(S)$ be the exponential normal map defined by,

$$E(t, x) = \exp tN_x .$$

Let $\gamma_x : [0, \infty) \to M$ be the normal geodesic defined by,

$$\gamma_x(t) = E(t, x).$$

Theorem C is an immediate consequence of the following two lemmas.

LEMMA 3.1. *If for each $x \in S$, $\gamma_x$ is an S-ray contained in $I^-(\gamma)$ then E is an isometry of $([0, \infty) \times S, -dt^2 \oplus h)$ onto $J^+(S)$, where h is the induced metric on S.*

LEMMA 3.2. *If $\gamma_x$ is an S-ray contained in $I^-(\gamma)$ then there exists a neighborhood V of x in S such that for each $y \in V$, $\gamma_y$ is an S-ray contained in $I^-(\gamma)$.*

*Proof of Lemma 3.1.* The proof is fairly standard. Let $U = E([0, \infty) \times S)$. Since each $\gamma_x$ is an $S$-ray, there can be no focal points to $S$ along any future directed normal geodesic. Furthermore, no two normal geodesics can intersect. It follows that $U$ is open and $E : [0, \infty) \times S \to U$ is a diffeomorphism.

Let $u$ be the unit tangent vector field to the normal geodesics, $u = E_*(\partial/\partial t)$. $u$ obeys the Raychaudhuri equation (see e.g. [HE]) for an irrotational geodesic vector field,

$$(3.1)\qquad u(\Theta) = -\operatorname{Ric}(u, u) - |\nabla u|^2,$$

where $\Theta = \operatorname{div}(u)$. By the Schwarz inequality we have,

$$\Theta \leqslant (n-1)|\nabla u|^2,$$

which when used together with the curvature assumption in (3.1) gives

$$(3.2)\qquad u(\Theta) + (n-1)^{-1} \cdot \Theta \leqslant 0.$$

Since $S$ is maximal, $\Theta = 0$ along $S$. Then (3.2) implies that $\Theta \leqslant 0$ on $U$. If $\Theta < 0$ at some point $p \in U$ then from (3.2) it follows that $\Theta \downarrow -\infty$ along the $S$-ray $\sigma$ through $p$ in a finite proper time, which implies that there exists a focal point to $S$ along $\sigma$. Thus we must have $\Theta \equiv 0$ on $U$, and hence from equation (3.1), $u$ is parallel on $U$. It follows that $E : [0, \infty) \times S \to U$ is an isometry.

It remains to show that $U = J^+(S)$. The inclusion $U \subset J^+(S)$ is trivial. If $S$ is geodesically complete then using the completeness of $S$ and the product structure of $U \approx [0, \infty) \times S$, one easily shows that every future directed causal curve starting at a point of $S$ cannot leave $U$. Hence, $J^+(S) \subset U$. To complete the proof of the lemma we show that if $S$ is future causally complete then it is geodesically complete.

Let $Q = I^-(\gamma) \cap J^+(S)$. We claim that $U = Q$. Since, by assumption, $\gamma_x \subset \subset I^-(\gamma)$ for all $x \in S$ we have $U \subset Q$. To show that $Q \subset U$, it suffices to show that $Q \subset \operatorname{int} D^+(S)$, because from each point of $\operatorname{int} D^+(S)$ there exists a maximal timelike geodesic to $S$ which meets $S$ orthogonally. If $Q$ is not contained in $\operatorname{int} D^+(S)$, then there exists a point $q \in H^+(S) \cap Q$. Let $\eta$ be a past inextendible null geodesic generator of $H^+(S)$ with future end point $q$. Since, by assumption, $S$ is future causally complete, the closure of $J^-(q) \cap S$, call it $A$, is compact. Since $J^+(A) \cap J^-(q) \subset Q$ and $Q$ is globally hyperbolic, one easily adapts Proposition 6.6.1. and its corollary in [HE] to show that $J^+(A) \cap J^-(q)$ is compact. Hence, $\eta$ is imprisoned in a compact set contained in $Q$, which contradicts the strong causality of $Q$. Thus, $U = Q$ and, in particular, $U$ is globally hyperbolic. It then follows from (a slight modification of) Theorem 2.5.3 in Beem and Ehrlich [BE] that $S$ is geodesically complete. ▪

We now proceed to the

*Proof of Lemma 3.2.* Let $\sigma$ be the $S$-ray $\gamma_x$, and set $I(\sigma) = I^-(\sigma) \cap I^+(\sigma)$. The proof involves an analysis of the Lorentzian Busemann function $b : I(\sigma) \to \mathbb{R}$ associated to $\sigma$. For each $r > 0$, define the function $b_r : M \to \mathbb{R}$ by,

$$b_r(x) = r - d(x, \sigma(r)).$$

For $x \in I^+(\sigma(0)) \cap I^-(\sigma(r))$, $b_r(x)$ is decreasing in $r$ and bounded below by $d(\sigma(0), x)$. Thus, $\lim_{r\to\infty} b_r(x)$ exists and, by definition, is $b(x)$.

We consider some properties of $b$ which are valid near the ray $\sigma$. An open set $U \subset I(\sigma)$ is said to be *nice* (with respect to $\sigma$) if there exist constants $K > 0$ and $T > 0$ such that for each $q \in U$ and $r > T$, any maximal unit speed geodesic segment $\alpha$ from $q$ to $\sigma(r)$ satisfies,

$$g_0(\alpha'(0), \alpha'(0)) \leqslant K,$$

where $g_0$ is some fixed Riemannian metric on $M$. We summarize some facts concerning nice neighborhoods.

1. For each $t > 0$, $\sigma(t)$ is contained in a nice neighborhood.

2. Asymptotes to $\sigma$ from points in nice neighborhoods are timelike.

3. $\{b_r\}$ converges locally uniformly to $b$ on nice neighborhoods, and hence $b$ is continuous on nice neighborhoods.

Properties 1 and 3 are proved in [E], and property 2 is a simple consequence of property 1. ▪

The proof of Lemma 3.2 relies heavily on the following result which is proved in [G3].

LEMMA 3.3. *Assume $M$ obeys the energy condition,* Ric$(X, X) \geqslant 0$ *for all $X$ timelike. Let $\Sigma$ be a connected smooth spacelike hypersurface contained in a sufficiently small nice neighborhood of $\sigma(t)$, $t > 0$. Assume the mean curvature of $\Sigma$ is nonnegative, $H_\Sigma \geqslant 0$. If $b$ achieves a minimum along $\Sigma$ then $b$ is constant along $\Sigma$.*

The mean curvature of $\Sigma$ is defined with respect to the future directed normal along $\Sigma$. We use the sign convention in which $H_\Sigma > 0$ corresponds to mean contraction of $\Sigma$. Although it is assumed in [G3] that $M$ is globally hyperbolic it is sufficient for the proof of Lemma 3.3 that $I(\sigma)$ be globally hyperbolic.

We will consider the Busemann function $b$ restricted to level sets of the Lorentzian distance function to $S$, $\delta : J^+(S) \to [0, \infty]$, defined by,

$$\delta(q) = d(S, q) = \sup_{y \in S} d(y, q).$$

The reverse triangle inequality implies,

$$\delta(q) \geqslant \delta(p) + d(p, q), \quad \forall p, q \in J^+(S),\ p \leqslant q. \tag{3.3}$$

By setting $q = \sigma(r)$ in (3.3) and using the fact that $\sigma$ is an $S$-ray we obtain,

$$b \geqslant \delta \text{ on } I(\sigma), \tag{3.4}$$

with equality holding along $\sigma$.

Suppose $\alpha : [0, \infty) \to M$ is a timelike geodesic asymptote to $\sigma$ emanating from a point in $I(\sigma)$. It can be shown that (see [E] or [G3]),

$$b(\alpha(t)) = t + b(\alpha(0)), \ \forall\, t \in [0, \infty).$$

Using this together with (3.3) and (3.4) gives,

$$t + \delta(\alpha(0)) \leqslant \delta(\alpha(t)) \leqslant t + b(\alpha(0)).$$

Thus, provided $\delta(\alpha(0)) = b(\alpha(0))$, we have,

$$\delta(\alpha(t)) = t + \delta(\alpha(0)), \ \forall t \in [0, \infty). \tag{3.5}$$

Fix $a > 0$. Near $S$, $\delta$ just measures proper time along the future-directed normal geodesics to $S$. Thus, for $a$ sufficiently small, there exists a neighborhood $U$ of $\sigma([0, a))$ such that $\delta\,|_U$ is smooth and $\nabla\delta$ is timelike. Then, for each $t \in [0, a]$,

$$\Sigma_t = \{q \in U: \delta\,(q) = t)\}$$

is a smooth spacelike hypersurface. Moreover, we can assume there is a neighborhood $V$ of $x$ in $S$ such that $\Sigma_t = E(\{t\} \times V)$ for all $t \in [0, a]$. It is well-known that $H_t = \operatorname{div}(\nabla\delta)\,|_{\Sigma_t}$ is the mean curvature of $\Sigma_t$ and obeys,

$$\partial H_t/\partial t = \operatorname{Ric}(\nabla\delta, \nabla\delta) + |\operatorname{Hess}\delta|^2, \tag{3.6}$$

(compare with (3.1)). Let $\Sigma = \Sigma_a$. By choosing $V$ sufficiently small we can assume that $\Sigma$ is contained in a sufficiently small (in the sense of Lemma 3.3) nice neighborhood of $\sigma(a)$. Since $H_0 = 0$, (3.6) implies that the mean curvature of $\Sigma$ is nonnegative. From (3.4) we have $b \geqslant a$ on $\Sigma$ and $b(\sigma(a)) = a$. Thus, according to Lemma 3.3,

$$b = a \ \text{along}\ \Sigma. \tag{3.7}$$

For each $y \in V$, there exists a timelike geodesic asymptote $\alpha_y : [0, \infty) \to M$ to $\sigma$ from $\gamma_y(a) \in \Sigma$. Consider the (possibly) broken geodesic $\alpha : [0, \infty) \to M$ defined by,

$$\alpha(t) = \begin{cases} \gamma_y(t), & 0 < t \leqslant a \\ \alpha_y(t-a), & a \leqslant t < \infty. \end{cases}$$

For $t \in [0, a]$, $\delta(\alpha(t)) = \delta(\gamma_y(t)) = t$. For $t \in [a, \infty)$, $\delta(\alpha(t)) = \delta(\alpha_y(t-a)) = (t-a) + a = t$, by (3.5) and (3.7). It follows that $\alpha$ is an $S$-ray, and hence that $\alpha = \gamma_y$. This concludes the proof of Lemma 3.2, and in turn the proof of Theorem C. ∎

Theorem C can now be used to establish Theorem B.

*Proof of Theorem B.* Suppose $H^+(S) \neq \phi$. By Lemma 2.1, there exists a future inextendible $S$-ray $\gamma$ contained in $D^+(S) \cap I^-(p)$ for some $p \in H^+(S)$. Theorem C implies that $J^+(S)$ is isometric to the Lorentzian product $[0, \infty) \times S$. However, it follows easily from the product structure of $J^+(S)$ that $\gamma$ cannot be contained in the past of any point in $J^+(S)$. Thus, $H^+(S) = \phi$. ∎

The following corollary improves Corollary 1 in [B] (compare also Theorem 3(c) in [MT]), by removing the assumption of global hyperbolicity.

COROLLARY 3.4. *Let $M$ be a noncompact space-time which satisfies the strong energy condition and contains a compact smooth spacelike hypersurface $S$ of constant mean curvature. Then $M$ is either timelike geodesically incomplete or splits as in Theorem 2.5.*

*Proof.* Suppose $M$ is timelike geodesically complete. Standard singularity theory (see [HE]) forces $S$ to be maximal. Just as in the proof fo Theorem 2.5, the assumption that $S$ is not acausal leads to the conclusion that $M$ is compact. Hence, we can assume that $S$ is acausal. Then Theorem B and its time-dual imply that $S$ is Cauchy, and in particular that $M = J^-(S) \cup J^+(S)$. Using standard arguments to produce future and past $S$-rays, Theorem Cand its time-dual can then be applied to obtain the desired splitting.

We remark in closing that by suitably modifying the proof of Theorem C one can obtain a Riemannian analogue of this result.

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